Boundary control of linearized Saint-Venant equations oscillating modes

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Abstract

The paper investigates the control of oscillating modes occurring in open-channels, due to the reflection of propagating waves on the boundaries. These modes are well represented by linearized Saint-Venant equations, a set of hyperbolic partial differential equations which describe the dynamics of one-dimensional open-channel flow around a given stationary regime. We use a distributed transfer function approach to compute a dynamic boundary controller that cancels the oscillating modes over all the canal pool. This result is recovered with a Riemann invariants approach in the case of a frictionless horizontal canal pool. The effect of a proportional boundary control on the poles of the transfer matrix is then characterized by a root locus, and we derive an asymptotic result for high frequencies closed-loop poles.

Key words: Open-channel system, Saint-Venant model, Frequency response; Root locus; Riemann invariants; Impedance matching; Water management

1 INTRODUCTION

The Saint-Venant equations describe the dynamics of open-channel hydraulic systems (e.g., rivers, irrigation and drainage canals or sewers) assuming one-dimensional flow. Firstly stated in 1871, these nonlinear partial differential equations have been widely used by hydraulic engineers in their numerical models \cite{2}. Many authors contributed on the control of open-channel hydraulic systems represented by Saint-Venant equations. The contributions range from classical linear control methods such as PI control, to $H_{\infty}$ robust control \cite{9} or $\ell_1$ control \cite{10}. A recent approach tried to take into account the distributed feature of the system by a Riemann invariants approach \cite{3}. A recent overview is given in \cite{4}.

To the best of our knowledge, no references deal with the problem of controlling the oscillating modes that appear on some types of canals, typically small canal pools. These modes can lead to water level oscillations and may provoke overtopping, which is highly undesirable for irrigation canals.

The objective of the paper is to investigate linearized Saint-Venant equations modes and their control. A reasonable physical intuition leads to think that the oscillations are generated by the reflection of propagating waves at the boundaries. We present three different approaches to achieve a boundary controller which does not reflect the propagating wave:

- First with an impedance matching method based on the distributed transfer matrix of Saint-Venant equations, leading to a dynamic boundary controller that cancels the oscillating modes over all the canal pool.
- Second with a Riemann invariant approach for the special case of a rectangular horizontal frictionless canal, leading to a proportional boundary controller.
- Third with a root locus technique, to investigate the effect of a static boundary controller on the closed-loop poles.

These three approaches are shown to be closely connected, and lead to an elegant solution for the damping of these oscillations.

The paper is organized as follows: the linearized Saint-Venant equations and the associated distributed transfer matrix are stated in Section 2. The main results of the paper are given in Section 3 with the dynamic controller

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and the Riemann invariants approach, and in Section 4, with the root locus technique.

2 SAINT-VENANT TRANSFER MATRIX

2.1 Linearized Saint-Venant equations

We consider in the paper a canal pool of length $X$ with prismatic and uniform geometry along $x$. The Saint-Venant equations are first order hyperbolic nonlinear partial differential equations given by [2, p. 15]:

\[
\begin{align*}
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + \frac{\partial Q^2/A}{\partial x} + gA\frac{\partial Y}{\partial x} &= gA\left(\frac{S_b - Q^2/2A^{4/3}}{A^{10/3}}\right)
\end{align*}
\]

(1)

(2)

where $A(x, t)$ is the wetted area ($m^2$), $Q(x, t)$ the discharge ($m^3/s$) across section $A$, $V(x, t)$ the average velocity ($m/s$) in section $A$, $Y(x, t)$ the water depth (m), $S_b$ the bed slope, $n$ the roughness coefficient ($sm^{-1/3}$), $P(x, t)$ the wetted perimeter (m) and $g$ the gravitational acceleration ($m/s^2$) (see Fig. 1).

We consider small variations of discharge $q(x, t)$ and water depth $y(x, t)$ around uniform stationary values $Q_0$ and $Y_0$. Let $F_0$ denote the Froude number $F_0 = V_0/C_0$ with $V_0$ the average velocity ($m/s$) and $C_0 = \sqrt{gA_0/T_0}$ the wave celerity ($m/s$). $T_0$ is the water surface top width (m). Throughout the paper, the flow is assumed to be subcritical, i.e. $F_0 < 1$.

Linearizing the Saint-Venant equations around these stationary values ($Q_0, Y_0$) leads to [8] for details:

\[
\begin{align*}
T_0\frac{\partial q}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + 2V_0\frac{\partial q}{\partial x} + (C_0^2 - V_0^2)\frac{\partial q}{\partial x} &= qT_0(1 + \kappa_0)S_0y - \frac{2gS_0}{V_0}q
\end{align*}
\]

(3)

(4)

\[\kappa_0 = \frac{7}{3} - \frac{4A_0}{3S_0T_0}\frac{dP_0}{dT} \] is a coefficient dependent on the form of the section.

The boundary conditions are the upstream and downstream discharges $q(0, t)$ and $q(X, t)$. We first assume that the control action variable is the downstream discharge $q(X, t)$ and that the measured variable is the downstream water level $y(X, t)$.

The paper is illustrated on a canal with a trapezoidal cross section of bottom width 0.18 m and sides slope 1:0.15 (Y:H). The considered pool is 75 m long, with a bottom slope of $1.5 \times 10^{-3}$ and Manning friction coefficient of 0.016 $sm^{-1/3}$, and the uniform regime corresponds to a discharge of $Q_0 = 601/s$ and a water depth of $Y_0 = 0.56$ m.

2.2 Saint-Venant distributed transfer matrix

Applying Laplace transform to the linear partial differential equations (3–4) results in a system of Ordinary Differential Equations (ODE) in the variable $s$, parameterized by the Laplace variable $s$:

\[
\begin{pmatrix}
\frac{d}{dx}(q(x)) \\
(2 + C_0^2)\frac{d}{dx}(y(x))
\end{pmatrix}
= A_0
\begin{pmatrix}
q(x) \\
y(x)
\end{pmatrix}
\]

(5)

with $A_0 = \begin{pmatrix} 0 & -T_0 s \\ -V_0 s + g(1 + \kappa_0)S_0 & C_0(1 - F_0^2) \end{pmatrix}$.

Solving the ODE (5) leads to the open-loop Saint-Venant distributed transfer matrix relating the water depth $y(x, s)$ and the discharge $q(x, s)$ at any point $x$ in the canal pool to the upstream and downstream discharges (see [6] for details):

\[
\begin{pmatrix}
y(x, s) \\
q(x, s)
\end{pmatrix}
= \lambda_1(s)
\begin{pmatrix}
1 \\
0
\end{pmatrix}
+ \lambda_2(s)
\begin{pmatrix}
1 \\
-1
\end{pmatrix}\sqrt{s}
\]

(6)

where $\lambda_1$ and $\lambda_2$ are the eigenvalues of matrix $A_0$.

\[
\lambda_i(s) = \frac{1}{C_0(1 - F_0^2)} \left[ F_0 s + \frac{gS_0(1 + \kappa_0)}{2C_0} + (-1)^i \sqrt{s} \right]
\]

(7)

with $\sqrt{s} = s^2 + \frac{gS_0(1 + \kappa_0 + F_0^2)}{2C_0} s + \frac{g^2 S_0^2(1 + \kappa_0)^2}{4C_0^2}$.

The oscillating modes are linked to the poles of the Saint-Venant transfer matrix. These poles are obtained as the solutions of $T_0 s e^{\lambda(s)X} - \lambda(s)X = 0$. There is a pole in zero (the canal pool acts as an integrator) and the other poles verify the following equation:

\[
x^2 + \frac{2gS_0}{V_0} \left(1 + \frac{\kappa_0 - 1}{2} F_0^2\right) + \frac{g^2(1 + \kappa_0)^2 S_0^2}{4C_0^2} + \frac{\kappa_0^2 C_0^2(1 - F_0^2)}{X^2} = 0
\]

with $k \in \mathbb{N}^*$ (the pole obtained for $k = 0$ simplifies with a zero).

The poles $p_k$ are then given by:

\[
p_k = \frac{gS_0}{V_0} \left[-1 - \frac{\kappa_0 - 1}{2} F_0^2 \pm (1 - F_0^2) \sqrt{\Delta(k)}\right]
\]

(8)
with \( \Delta(k) = \frac{1 - i(k_0 - 1) F_0^2}{1 - F_0^2} - k^2 \tau^2 \xi^2 \eta^2 \). Let \( k_m \in \mathbb{N}^* \) be the greatest integer such that \( \Delta(k_m) \geq 0 \). Then the poles obtained for \( 0 < k \leq k_m \) are negative real, and those obtained for \( k > k_m \) are complex conjugate, with a constant real part (they are located on a vertical line in the left half plane). We focus in the paper on canal pools with a dominant oscillating behavior, i.e. corresponding to \( \Delta(1) < 0 \). Let us note that in the case of zero slope and frictionless canal, the poles are located on the imaginary axis.

Let \( k_h > k_m \) denote the smallest integer such that \( \forall k > k_h, |p_k| \geq \tau_h = \frac{2 \alpha S_0}{\tau_0^2} (1 + \frac{\alpha - 1}{2} F_0^2) \). One then has the following

**Proposition 1** For \( k \gg k_h \), the open-loop poles of Saint-Venant transfer matrix (6) can be approximated by:

\[
p_k \approx \frac{(1 + \alpha_2) X}{\tau_1 + \tau_2} \pm \frac{2 j k \pi}{\tau_1 + \tau_2}
\]

with \( \alpha_1 = \frac{T_0 S_0(2 - (\alpha_0 - 1) F_0)}{2 A_0 F_0(1 + F_0)} \), \( \alpha_2 = \frac{T_0 S_0(2 + (\alpha_0 - 1) F_0)}{2 A_0 F_0(1 - F_0)} \) and \( \tau_1 = \frac{X}{\tau_0 + V_0} \), the delay for downstream propagation, \( \tau_2 = \frac{X}{\tau_0 - V_0} \), the delay for upstream propagation.

**Proof:** Eq. (9) is obtained from straightforward manipulations of Eq. (8) for \( k \gg k_h \).

This result shows that for high frequencies the poles of Saint Venant transfer matrix are close to the ones of the following damped wave equation:

\[
\frac{\partial}{\partial x} + \alpha_1 \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} - \alpha_2 \frac{\partial}{\partial t} \right) q = 0
\]

with boundary conditions \( q(0, t) \) and \( q(X, t) \). Using Laplace transform, this equation reduces to an ODE in \( x \), with eigenvalues equal to \( -\alpha_1 - \frac{2 j k \pi}{\tau_0} \) and \( \alpha_2 + \frac{2 j k \pi}{\tau_0} \). The obtained transfer function has for denominator \( D(s) = 1 - e^{-(\alpha_1 + \alpha_2)X - (\tau_1 + \tau_2)s} \), whose roots coincide with the poles approximation (9). This shows that the oscillating modes correspond to the interaction of two gravity waves, one travelling downstream at speed \( V_0 + C_0 \) with attenuation factor \( \alpha_1 \), and one travelling upstream at speed \( C_0 - V_0 \) with attenuation factor \( \alpha_2 \).

3 EXACT CANCELLATION OF OSCILLATING MODES

3.1 Dynamic boundary controller

We show in this section that it is possible to cancel oscillating modes over all the canal pool by using a dynamic boundary controller.

**Theorem 1** With a downstream boundary control \( q(X, s) = k_u(s) y(X, s) \)  defined by

\[
k_u(s) = -\frac{T_0 s}{\lambda_1(s)} \tag{10}
\]

the canal pool represented by the distributed transfer matrix (6) has no oscillating modes.

**Proof:** Connecting the open-loop distributed transfer matrix of Eq. (6) with a downstream boundary controller \( q(X, s) = k_u(s) y(X, s) \) leads to the closed-loop distributed transfer matrix:

\[
\begin{pmatrix}
y(x, s) \\
q(x, s)
\end{pmatrix} =
\begin{pmatrix}
G_{k_u}^{(1)}(x, s) \\
G_{k_u}^{(2)}(x, s)
\end{pmatrix} q(0, s) \tag{11}
\]

with \( G_{k_u}^{(1)}(x, s) \) and \( G_{k_u}^{(2)}(x, s) \) given by:

\[
G_{k_u}^{(1)} = \frac{\lambda_2 X (e^{\lambda_2 X} - e^{\lambda_1 X}) + \lambda_1 X (e^{\lambda_1 X} - e^{\lambda_2 X})}{\eta_0 (e^{\eta_1 X} - e^{\eta_2 X}) + \eta_1 X (e^{\eta_2 X} - e^{\eta_1 X})} \tag{12}
\]

The oscillating poles of the closed-loop system (11) are solutions of the following equation:

\[
e^{(\lambda_2(s) - \lambda_1(s)) X} - \frac{T_0 s + k_u(s) \lambda_1(s)}{T_0 s + k_u(s) \lambda_2(s)} = 0 \tag{13}
\]

With the controller \( k_u(s) \), we get:

\[
e^{(\lambda_2(s) - \lambda_1(s)) X} = 0
\]

which has no finite solution and thus the system has no oscillating modes.

**Remark 1** This result is similar to the classical concept of "impedance matching" for electrical networks [1]. Indeed, with the controller \( k_u(s) \) given by Eq. (10), the distributed transfer functions are given by \( G_{k_u}^{(1)}(x, s) = -\frac{\lambda_1(s) e^{\lambda_1(s) X}}{T_0 s} \) and \( G_{k_u}^{(2)}(x, s) = e^{\lambda_1(s) X} \), and thus only the downstream propagating waves remain.

The Bode diagram of \( k_u(s) \) is depicted in Fig. 2 for the considered canal pool. This stable, infinite-dimensional controller strongly looks like a lead-lag filter. It is not strictly proper, since it has a constant gain in high frequencies. The optimal dynamic controller (10) can be interpreted as a non-reflexive downstream boundary condition. With this controller, the canal behaves as if it was semi-infinite, i.e. the waves propagating downstream do not reflect on the downstream boundary and the oscillating modes then disappear. This idea can also be applied using Riemann invariants.
3.2 Riemann invariants approach

We restrict the study in this section to the special case of a rectangular horizontal frictionless canal, for which the Riemann invariants have closed-form expressions:

\[ J_+ (x, t) = V (x, t) + 2C (x, t) \]
\[ J_- (x, t) = V (x, t) - 2C (x, t) \]

\( J_+ \) and \( J_- \) are called the Riemann invariants of Eqs. (1–2) and are easily shown to be constant along the characteristics curves defined as:

\[ \frac{dx}{dt} = V (x, t) + C (x, t) \]
\[ \frac{dx}{dt} = V (x, t) - C (x, t) \]

A way to eliminate the oscillating modes is to suppress the reflection of downstream propagating waves on the downstream boundary, i.e. to ensure that a perturbation reaching the boundary does not generate an upstream propagating perturbation. This can be done by specifying a boundary controller such that the Riemann invariant at the boundary \( J_- (X, t) \) remains constant for any \( t > 0 \):

\[ J_- (X, t) = V (X, t) - 2C (X, t) = V_0 - 2C_0 \]

(14)

where \( V_0 \) and \( C_0 \) correspond to the initial condition of the canal pool.

Then, as noted by [5], the Riemann invariant \( J_- (x, t) \) is constant for all \( x \in [0, X] \) and all \( t \geq \tau_1 + \tau_2 \). In this case, the channel behaves as if it was semi-infinite, since all waves arriving at the downstream boundary “cross” it without reflection. Therefore, no oscillating modes can occur in the canal pool. The corresponding downstream boundary controller can be obtained by linearizing relation (14) around \( V_0 \) and \( C_0 \):

\[ v (X, t) - 2c (X, t) = 0 \]

(15)

with \( v (X, t) = \frac{1}{T_0 V_0} q (X, t) - \frac{Q}{T_0 V_0} y (X, t) \) and \( c (X, t) = \frac{C_0}{2T_0} y (X, t) \). Eq. (15) is then equivalent to:

\[ q (X, t) = T_0 (C_0 + V_0) y (X, t) \]

(16)

Therefore, a proportional boundary controller of gain \( T_0 (C_0 + V_0) \) linking the discharge to the water elevation eliminates the oscillating modes in the special case of a rectangular horizontal frictionless canal pool. This result is recovered with the transfer function approach, since in this case, the first eigenvalue is equal to \( \lambda_1 (s) = -s/(C_0 + V_0) \), and the optimal controller \( k_\omega (s) \) given by Eq. (10) becomes a static controller:

\[ k_\omega (s) = T_0 (C_0 + V_0) \]

Moreover, this gain corresponds to the high frequency asymptotic value of the optimal controller given by

\[ \lim_{\omega \to \infty} |k_\omega (j \omega)| = T_0 (C_0 + V_0) \]

(17)

4 PRACTICAL DAMPING OF OSCILLATING MODES

4.1 Implementation aspects

We now consider the implementation of a boundary controller on a realistic canal with slope and friction. Since it is not strictly proper, the optimal controller \( k_\omega (s) \) seems difficult to implement on a real canal, because of the actuator bandwidth limitation. This also applies to the proportional controller obtained in the zero-slope case, as long as it is implemented with a motorized gate. However, a constant gain in high frequency can be implemented by using a structural property of hydraulic structures such as gates (or weirs). A gate is usually described by a static nonlinear relationship \( Q = f (Y, W) \), where \( Q \) is the discharge, \( Y \) the water depth and \( W \) the gate opening. A classical expression of function \( f \) is \( f (Y, W) = aW \sqrt{Y} \) with \( a \) a constant coefficient depending on the gate characteristics. When considering small variations around stationary values, one gets the linearized equation \( q = k_w y + k_u w \), with \( k_u = \frac{dr}{dv} \) and \( k_w = \frac{dW}{dv} \).

Since the gate opening \( w \) is typically controlled by an electrical actuator with finite bandwidth, high frequency control cannot be achieved by “dynamic” feedback through \( k_w \). In general, it is thus only possible to use the structural static feedback \( k_u \) that directly links the water level \( y \) to the discharge \( q \) to achieve a high frequency control politics.

The question that arises is then: how to choose \( k_u \)? Analyzing the Bode diagram of Fig. 2 at a frequency corresponding to the first oscillating mode \( (\omega_1 = 10^{-1} \text{ rad/s}) \) shows that the amplitude of the optimal controller has almost reached its asymptotic value given by Eq. (17). We show below using a root-locus technique that this value leads to the best damping of oscillating modes.
4.2 Static boundary controller

According to Eq. (11), the closed-loop system poles for a static boundary proportional controller of gain $k_u \in \mathbb{R}^+$ are given by the solutions of the following equation:

$$
\psi(s) := e^{(\lambda_2(s) - \lambda_1(s))X} - \frac{T_0s + k_u\lambda_1(s)}{T_0s + k_u\lambda_2(s)} = 0 \quad (18)
$$

This equation has no closed-form solution in general. Numerical resolution for different values of $k_u$ leads to the root locus depicted in Fig. 3. For $k_u = +\infty$, the poles coincide with the open-loop zeros of the Saint-Venant transfer matrix. We observe that the closed-loop poles negative real parts reach a minimum for the optimal static controller value and that the modes damping increases with the frequency (i.e., higher frequency modes are more damped than low frequency modes). The following proposition provides a closed-form result explaining this behavior for high frequency poles:

**Proposition 2** When $|s| \gg r_h$, the solutions of Eq. (18) tend asymptotically towards

$$
\tilde{p}_k = -\frac{(\alpha_1 + \alpha_2)X}{\tau_1 + \tau_2} - \frac{1}{\tau_1 + \tau_2} \log \left( \frac{T_0X + k_u\tau_2}{T_0X - k_u\tau_1} \right) + \frac{2jk\pi}{\tau_1 + \tau_2} \quad (19)
$$

and the approximation error is at the first order given by:

$$
p_k \approx \frac{\psi(\tilde{p}_k)}{\psi'(\tilde{p}_k)} \quad (20)
$$

**Proof:** The proof is omitted for lack of space. See the technical report [7] for details.

Eq. (19) recovers the open-loop poles approximation given by (9) when $k_u = 0$. When $k_u$ increases, the poles real part diminishes towards $-\infty$ for $k_u < T_0(C_0 + V_0)$. Then, it increases when $k_u > T_0(C_0 + V_0)$, to finally tend towards $-\frac{(\alpha_1 + \alpha_2)X}{\tau_1 + \tau_2} \log \left( \frac{2}{2e} \right)$ when $k_u \to \infty$, which corresponds to the high frequency approximation of the open-loop zeros of the Saint-Venant transfer function. Zeros have a real part smaller than the one of the open-loop poles (because $\tau_1 < \tau_2$) and their imaginary part is given by $\pm(2k+1)\pi/(\tau_1 + \tau_2)$, because the complex logarithm verifies $\log(-1) = \pm j\pi$.

The pool behavior in the three extreme situations is evaluated by the maximum amplitude of $G^{(1)}_{k_u}(j\omega, x)$ for $\omega \in [\omega_r, +\infty)$, where $\omega_r$ is the frequency of the first oscillating mode. Fig. 4 depicts this variable along the longitudinal abscissa $x$, with different gains: $k_u = 0$, $k_u = +\infty$, $k_u = T_0(C_0 + V_0)$ and $k_u = k_u^*(s)$, the optimal dynamic controller. The optimal dynamic controller $k_u^*(s)$ gives the best overall performance, since it suppresses the oscillating modes over all the canal pool, but even the proportional controller enables to remarkably dampen the modes, compared to the other cases.

5 CONCLUSION

The paper provides a detailed study of the boundary control of oscillating modes for linearized Saint-Venant equations. Three methods are proposed to cancel or dampen the modes due to the reflection of propagating waves on the boundary: first an impedance matching method based on the distributed transfer matrix, second a Riemann invariants approach in the case of a frictionless horizontal canal pool, and third a root locus method, for which we derive an asymptotic result for high frequencies closed-loop poles. These results could certainly be extended to more general types of hyperbolic conservation laws. Future works will consider the behavior of low frequency modes and the internal stability of the boundary controlled hyperbolic system.
References


