Modelling and Robust Control of a Dam-River System

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ABSTRACT
The paper deals with the modelling and the automatic control of a dam-river system, where the action variable is the upstream flow rate and the controlled variable the downstream flow rate. The system is modeled with a linear model (second order transfer function with delay). Two control methods (pole placement and Smith predictor) are compared in terms of performance and robustness. The pole placement is done on the sampled model, whereas the Smith predictor is based on the continuous model. Robustness is estimated with the use of margins, and also with the use of a bound on multiplicative uncertainty for variable reference discharges. Simulations are carried out on a nonlinear model of the river, and performance of both controllers are compared to the one of a continuous-time PID controller.

1. INTRODUCTION
The paper develops two classical SISO methods for the flow control of a dam-river system and compares their robustness to nonlinearities due to reference flow rate variations. A linear model derived from simplified Saint-Venant equations is used for controller design. The robustness of the design is evaluated using a bound on multiplicative uncertainty, designed to capture variations in the reference flow rate. A continuous-time PID controller is also designed and tuned following Haalma's rule [10], to compare its performances to the one of both robust controllers.

The approach followed is a posteriori for the pole placement method, as the robustness of the controller is evaluated after design. Design parameters $T_c$ and $T_f$ following de Lamnait [1] are then tuned in order to satisfy robustness margins specifications. For the Smith Predictor (SP) method, it is an a priori approach, as the performance and robustness requirements are imposed to the controller before the design. The Smith Predictor is written in the form of an Internal Model Controller (IMC). The design is facilitated by the fact that a single design parameter, the filter coefficient $\lambda$, has to be determined.

2. SYSTEM DESCRIPTION AND DESIGN GOALS
2.1. Presentation of the system
The irrigation system considered uses natural rivers to convey water released from the upstream dam to consumption places. Farmers can pump water in the river when they need it without having to ask for it (it is an on-demand management). A (simplified) system considered is depicted in Figure 1: a dam and one river reach with a measuring station at its downstream end, and a pumping station just upstream. Pumping stations are in fact distributed along the river. This is taken into account during the identification process [2], but for simplicity, it is supposed that all pumping stations can be aggregated into one at the end of the reach. As this discharge $Q_{out}$ is not measured and not controllable, it is considered as a perturbation, that has to be rejected.

![Figure 1: Simplified dam-river system](image)

The controlled variable is the flow rate at the downstream end of the river. The water elevation is not controlled, as the system is used mainly in summer for maize irrigation, when the flow rate is quite low. The control action variable is the upstream flow rate. It is therefore assumed that there is a local (slave) controller at the dam that acts on a gate such that the desired flow rate is delivered.

The control objectives are twofold:
- satisfy the water demand from farmers (i.e. the discharge $Q_{out}$);
- keep the flow rate at the downstream end of the reach close to a reference flow rate (target), defined for hygienic and ecological reasons.

The main problem encountered in such systems is the robustness to varying time delays. As already stated by Papageorgiou and Messmer [3], the dynamics of a river stretch
are nonlinear, and depend on the reference discharge (see Figure 2). Such time delay variations can destabilize a linear controller designed without taking them into account.

**Figure 2**: Positive and negative step responses for different reference discharges $Q_0$: 5 m$^3$/s (dashed), 1 m$^3$/s (continuous line), 0.5 m$^3$/s (dotted)

### 2.2. Modelling of the system and uncertainty description

Open channel flow are well described by Saint-Venant equations [4]:

$$
\begin{align*}
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= q_l \\
\frac{\partial Q}{\partial t} + \frac{\partial (Q^2/A)}{\partial x} + A \frac{\partial z}{\partial x} &= -A g S_f + k q_l V
\end{align*}
$$

(1)

$Q(x,t)$ is the discharge (m$^3$/s) across section $A$, $q_l(x,t)$ the lateral discharge (m$^2$/s), $q_l>0$: inflow, $q_l<0$: outflow), $A(x,z)$ the wetted area (m$^2$), $z(x,t)$ the absolute water surface elevation (m), $S_f(Q,z,x)$ the friction slope, $V(x,t)$ the mean velocity (m/s) in section $A$, and $g$ the gravitational acceleration (m/s$^2$), $k=0$ if $q_l>0$ and $k=1$ if $q_l<0$, considering that inflows are perpendicular to the flow, therefore not contributing to the momentum and outflow are parallel to the flow, diminishing the momentum.

Under simplifying hypotheses, the water elevation $z$ can be eliminated from Saint-Venant equations, leading to the Diffusive Wave equation [4]:

$$
\frac{\partial Q}{\partial t} + C(Q,z,x) \frac{\partial Q}{\partial x} + D(Q,z,x) \frac{\partial^2 Q}{\partial x^2} = 0
$$

(2)

with $Q(x,t)$ the discharge (m$^3$/s), $C(Q,z,x)$ the celerity (m/s), and $D(Q,z,x)$ the diffusion (m$^2$/s).

The following form is assumed for the celerity and diffusion coefficients:

$C(Q) = \alpha_C Q^{\beta_C}$, and $D(Q) = \alpha_D Q^{\beta_D}$.

The identification of the four parameters $\alpha_C$, $\beta_C$, $\alpha_D$, $\beta_D$ for a river reach proved to be efficient on simulated as well as on real data [2].

Linearizing equation (2) around a reference discharge $Q_0$ leads to the Huygens equation, which can be analytically identified to a second order plus delay transfer function [5, 6]:

$$
F(s) = \frac{\exp(-st)}{(1+sK_1)(1+sK_2)}
$$

(3)

The analytical identification process enables to express the coefficients of $F(s)$ in function of physical parameters, as the reference discharge $Q_0$.

The uncertainties due to variable reference discharges are represented as an output multiplicative uncertainty. This multiplicative uncertainty captures time delay as well as dynamics variations, which are due to the nonlinearity of the process.

For $Q_0 \in [Q_{\text{min}}, Q_{\text{max}}]$, the transfer function $F(s)$ is written as:

$$
F(s) = [1 + \Delta_m(s)] F_0(s)
$$

(4)

with $|\Delta_m(s)| \leq |F_0(s)| \forall s$.

$F_0(s)$ is the nominal model, used to design the controller.

**Figure 3**: Bound $\Delta_m$ on multiplicative uncertainty

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### 3. ROBUST CONTROL DESIGN

#### 3.1. Robustness evaluation

##### 3.1.1. Robustness margins

Consider the feedback system of Figure 4.

**Figure 4**: Feedback system

It gives the following relations:

$$
\begin{align*}
y &= S_y (r - b) \\
u &= K S_y (r - b - d)
\end{align*}
$$

with $S_y = \frac{1}{1 + L}$, $T_y = \frac{L}{1 + L}$.

$L = F K$ is the open loop transfer function, $S_y$ is the output-perturbation sensitivity function, and $T_y$ the complementary sensitivity function.

$T_y$ and $S_y$ are linked by the relation $S_y + T_y = 1$.

In the following, the definitions of classical robustness margins [7] are recalled, along with simple explanations of their physical meaning. These margins offer a simple way to evaluate the robustness of a controlled system, in terms of acceptable variations in gain, phase, or time delay.
The modulus margin $M_m$ is defined as the minimal distance of the Nyquist plot of $L$ to the point (-1,0):

$$M_m = \inf \{ |1+L(j\omega)|, \omega \in \mathbb{R} \}$$

Then:

$$M_m = |1+L(j\omega)|_{\min} = \frac{1}{|S_y(\omega)|_{\max}}$$

where $|S_y(\omega)|_{\max}$ represents the maximum of $|S_y(j\omega)|$ for $\omega \in \mathbb{R}$.

If the Nyquist plot of $L(j\omega)$ intersects the unit circle in more than one point, noted $\omega_c L(j\omega_c) = 1$ and the x-axis in more than one point, noted $\omega_c L(j\omega_c) = \pi/2$ (i.e. $\arg L(j\omega_c) = \pi - 2\pi$), the gain margin $M_g$ is defined as:

$$M_g = \min k(\theta_k), \quad \theta_k = \frac{1}{|L(j\omega_c)|}$$

If $K$ stabilizes $G_0$ with a gain margin $g$, then $K$ stabilizes $G$ of the form $K G_0$ for every $k$ between 0 and $g$.

The delay phase margin $M_{dp}$ is defined as:

$$M_{dp} = \min k(\theta_k), \arg L(j\omega_c) = \pi + \theta_k$$

The advance phase margin $M_{ap}$ is defined as:

$$M_{ap} = \min k(\theta_k), \arg L(j\omega_c) = \pi - \theta_k$$

If $K$ stabilizes $G_0$ with a delay (resp. advance) phase margin $\theta_c$, then $K$ stabilizes $G$ of the form $e^{j\tau} G_0$ (resp. $e^{j\tau} G_0$) for every $\tau$ between 0 and $\tau_c$.

The delay margin $M_d$ is defined as the maximum of the time delays $\tau$ such that the feedback system is stable for a perturbed process $R_{e} F$ ($F$ represents the delay operator), of transfer function $e^{j\tau S}$:

$$M_d = \min k \left( \frac{\phi_k}{\omega_c \tau_k} \right), \arg L(j\omega_c) = \pi + \phi_k$$

and the advance margin $M_a$:

$$M_a = \min k \left( \frac{\phi_k}{\omega_c \tau_k} \right), \arg L(j\omega_c) = \pi - \phi_k$$

Then, if $K$ stabilizes $G_0$ with a delay (resp. advance) margin $\tau_m$, then $K$ stabilizes $G$ of the form $e^{j\tau S} G_0$ (resp. $e^{j\tau S} G_0$) for every $\tau$ between 0 and $\tau_m$.

### 3.2. Robust continuous Smith predictor design

#### 3.2.1. Internal model control representation of the Smith predictor

The nominal transfer function $F_0$ is factored in two terms, $F_{M0}$ being the part without delay:

$$F_0(s) = \frac{\exp(-\tau_0)}{(1 + s K_1)(1 + s K_2)} = \frac{F_{M0}(s) \exp(-\tau_0)}{1 + s K_1(s)}$$

$$F_0(s) = \frac{F_{M0}(s) \exp(-\tau_0)}{1 + s K_1(s)}$$

#### 3.2.2. Robust stability and performance of the Smith predictor

**Robust stability:**

Using the IMC representation, the robust stability condition (10) becomes:

The system of Figure 5 is stable for multiplicative uncertainties $|\Delta L(j\omega)| \leq |L_m(\omega)|$ if:

- the nominal system is stable
- $|Q(j\omega)F_0(j\omega)| < |L_m(\omega)|^{-1}$ \forall\omega

**Robust performance:**

Nominal performance is specified with an $H_{\infty}$ constraint on nominal sensitivity function $S_y(\omega)$:

$$S_y(\omega) = \frac{1}{|L_y(\omega)|}$$

where $L_y$ is a weighting function for performance. For example, a choice of $L_y = 1 + 2\pi$, ensures that the sensitivity function $S_y$ stays below MP (see [9] for different choices of weighting functions $L_y$).

The performance is robust when the inequality is respected for $S_y(\omega)$.

Robust stability and performance can be combined in one inequality:

$$|T_y(j\omega)| L_m(\omega) + |[1 - T_y(j\omega)]/L_y(j\omega)| < 1$$

or:

$$\frac{|T_y(j\omega)|}{1 - |[1 - T_y(j\omega)]/L_y(j\omega)|} L_m(\omega) < 1$$

This condition can be checked by plotting frequency responses of both terms of the inequality (see Figure 6).

The advantage of the IMC parametrization is that the design of the controller results in the choice of a single design.
parameter, as the $H_2$-optimal controller $Q(s)$ can be calculated analytically, and a filter $f(s)$ is added to ensure robustness to the feedback system [9]:

$$f(s) = \frac{1}{(1 + \lambda s)^n}$$

$n$ is chosen to make $f(s)Q(s)$ proper, and $\lambda$ is the design parameter of the IMC controller.

![Frequency responses of $I_m(w)^{-1}$ (continuous line) and $1/(1 - T_{Vd}(w))w_{20}(w)$ (dashed), SP controller.](image)

Once the controller is designed, the robustness margins are easy to calculate using equations (5)-(9). The robustness margins obtained for the three controllers are given in Table 1.

<table>
<thead>
<tr>
<th>Controllers</th>
<th>SP</th>
<th>RST</th>
<th>PID</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain margin $(M_g)$</td>
<td>3.24</td>
<td>3.49</td>
<td>2.36</td>
</tr>
<tr>
<td>Delay margin $(M_d)$</td>
<td>19.6 h</td>
<td>16.4 h</td>
<td>9.2 h</td>
</tr>
<tr>
<td>Advance margin $(M_a)$</td>
<td>86.8 h</td>
<td>69.8 h</td>
<td>54.8 h</td>
</tr>
<tr>
<td>Modulus margin $(M_m)$</td>
<td>0.66</td>
<td>0.69</td>
<td>0.52</td>
</tr>
</tbody>
</table>

Table 1: Robustness margins for the three controllers

3.3. Robust digital pole placement design

3.3.1. Digital RST controller

With the transfer function of the system written as:

$$F^*(z) = \frac{az^{-1} + ez^{-2}}{1 - az^{-1} + bz^{-2}} \frac{B(z)}{A(z)},$$

the RST regulator is represented in Figure 7:

![Structure of the RST regulator](image)

The output $y$ is expressed as function of $d$, $y_C$ and $b$:

$$y = \frac{AS}{AS + BR} d + \frac{BT}{AS + BR} y_C - \frac{BR}{AS + BR} y$$

(Argument $z$ is omitted for readability).

Sensitivity functions are expressed as functions of polynomials $R$, $S$, $A$ and $B$:

$$S_y = \frac{AS}{AS + BR} \cdot T_y = \frac{BR}{AS + BR}$$

The open loop transfer function is $L = \frac{BR}{AS}$.

Stability of the closed loop system is ensured if the roots of $AS + BR$ are inside the unit circle.

3.3.2. Robust pole placement (de Larminat method) [11]

The method provides a way to place poles of a feedback system with two design parameters, $T_C$ and $T_F$. It is based on the following heuristic remark, valid for a stable process: the less the poles of the process are modified by the feedback, the more robust the closed loop system is with respect to model uncertainties.

The method proposes to place the poles of the feedback system from the poles of the nominal process:

- in continuous-time the poles of the process which are to the right of a straight line $x = -1/T_C$ are projected on this line.
- in discrete-time, the poles of the nominal system with modulus greater than $R_0 = \exp(-T_F/T_C)$ are projected on the circle of ray $R_0$, the others are left in place.

This gives the $n$ dominant poles ($n$ is the degree of $A(z^{-1})$), and the $n+1$ filter poles are determined with another parameter $T_F$, and a real pole is added at $\exp(-T_F/T_C)$.

The closed loop polynomial has $2n+1$ zeros, determined with 2 design parameters $T_C$ and $T_F$.

![Robust performance condition, RST controller.](image)

The RST controller is robust stable w.r.t. multiplicative uncertainties, but does not satisfy the robust performance condition, which is slightly violated (Figure 8).

3.4. PID controller design

Both controllers are compared to a continuous-time PID controller obtained with Haanman's method, suited for systems where dynamics are dominated by dead-time [10]. This is a loop shaping method where the desired loop transfer function $L_0$ is specified, and the controller transfer function is obtained as $K = \frac{L_0}{P}$, where $P$ is the plant transfer function.
Such an approach can give PID controllers provided that $L_0$ and $F$ are sufficiently simple.

For systems with a time delay $\tau$, Haalmans has suggested to choose $L_0(s) = \frac{2}{3s^2}e^{-\tau s}$.

The value $2/3$ was found by minimizing the mean square error for a step change in the set point. This choice gives a sensitivity $M_s = 1.9$ (or a modulus margin $M_m = 1/1.9 = 0.53$), and a delay margin:

$M_d = \frac{3\pi - 4}{4} \tau = 1.36 \tau$

For systems with the transfer function

$$F_p(s) = \frac{\exp(-s\tau)}{(1 + sK_0)(1 + sK_20)}$$

the controller obtained is a PID: $K(s) = K_p(1 + \frac{1}{sT_1} + sT_d)$

with the parameters:

$$K_p = \frac{2(K_10 + K_20)}{3\tau_0}, \quad T_1 = K_10 + K_20, \quad T_d = \frac{K_10 K_20}{K_10 + K_20}.$$

Such a controller cancels the poles of the system, which may lead to controllability problems if the process is lag dominated, which is usually not the case in dam-river systems.

![Figure 9: Robust performance condition, PID controller.](image)

The PID controller is robust stable w.r.t. multiplicative uncertainties, but does not satisfy the robust performance condition (Figure 9).

4. RESULTS OF SIMULATIONS AND DISCUSSION

4.1. Simulations

Results of simulations given by both controllers are compared to those given by a continuous-time PID controller, tuned following Haalmans method. Simulations are done on a nonlinear simulation model [2] solving equation (2). Three different scenarios are tested with feed-forward (FF) and feedback (FB) controllers:

- reference tracking around an average flow rate
- predicted disturbance rejection around an average flow rate
- unknown disturbance rejection around average and extreme flow rates.

![Figure 10: Unknown disturbance rejection around Q0 = 0.5 m³/s, SP controller](image)

![Figure 11: Unknown disturbance rejection around Q0 = 0.5 m³/s, RST controller](image)

![Figure 12: Unknown disturbance rejection around Q0 = 0.5 m³/s, PID controller](image)
4.2. Discussion

Results of simulation show that all controllers are stable, as suggested by the robustness analysis in the frequency domain. It should be mentioned that we have no theoretical insurance for this, as the stability is ensured by (10) for a set of linear models captured in the nominal model and the multiplicative uncertainty, but not for the nonlinear system.

The PID controller is the quicker to respond to unpredicted disturbances, but it is also the more oscillating. As such oscillating commands would not be allowed in reality, it would have to be filtered, therefore leading to slower responses.

The RST controller behaves well, although oscillating around $Q_0 = 0.5 \text{ m}^3/\text{s}$. The Smith Predictor is a little slower than the other two, but gives realistic commands for all operating points.

It is important to notice the fact that there is a trade-off between robustness and performance. A certain level of performance cannot be achieved without loosing some robustness (at least in the linear case), and respectively, to increase the robustness of a feedback loop, the performance requirements must be lowered.

More than the results of simulation, where different values of tuning parameters can be chosen depending on the requirements, the emphasis is made on the tuning methods, and the robustness analysis tools developed.

Both tuning methods (SP and RST) are easy to manage, and robustness can be checked a posteriori. The tuning parameters have a clear physical interpretation:

- the filter parameter $\lambda$ in the IMC design is called the robustness filter: the larger $\lambda$, the more robust the controlled system, but perturbations are rejected slowly.

- the robust pole placement method has two tuning parameters $T_C$ and $T_F$; $T_C$ is a control horizon, and $T_F$ a filtering horizon. The larger they are, the slower the controlled system will be. Their values can be adjusted by trial and error, using the pole location, or the value of robustness margins as a tuning criteria.

On the other hand, the PID controller does not offer clear tuning parameters, and the method may not be suited for other systems.

5. CONCLUSION

The paper presents some tools to check the robustness of feedback loops for an open channel hydraulic system. The tools are illustrated on two different approaches:

- an 'a priori' approach to design a robust continuous-time Smith predictor: the controller is designed taking into account the multiplicative uncertainty on the nominal model,

- an 'a posteriori' approach to design a robust discrete-time pole placement RST controller: the design parameters $T_C$ and $T_F$ are tuned by trial and error until satisfactory robustness margins are obtained.

Controllers proved to be robust and have a good performance, compared to a PID controller tuned with Haalmaan’s method, and both methods are suited for real-time applications.

6. REFERENCES


