Boundary control of hyperbolic conservation laws using a frequency domain approach

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Abstract

The paper uses a frequency domain method for boundary control of hyperbolic conservation laws. We show that the transfer function of the hyperbolic system belongs to the Callier-Desoer algebra, which opens the way of sound results, and in particular to the existence of necessary and sufficient condition for the closed loop stability and the use of Nyquist type test. We examine the link between input-output stability and exponential stability of the state. Specific results are then derived for the case of proportional boundary controllers. The results are illustrated in the case of boundary control of open channel flow.

Key words: Hyperbolic system; Frequency response; Water management; Exponential stability

1 INTRODUCTION

Hyperbolic conservation laws are derived from physics of distributed parameter systems. We deal in this paper with systems represented by hyperbolic conservations laws with an independent time variable $t \in [0, +\infty)$ and an independent space variable on a finite interval $x \in [0, L]$, for which we derive stabilizing boundary controllers using a frequency domain approach.

This work is motivated by the problem of controlling an open channel represented by Saint-Venant equations. These hyperbolic Partial Differential Equations (PDE) describe the dynamics of open channel hydraulic systems, e.g. rivers, irrigation or drainage canals, sewers, etc., assuming one dimensional flow.

Many authors contributed on the control of open channel hydraulic systems represented by Saint-Venant equations. The contributions range from classical monovariable control methods such as PI control [20,27] to multivariable LQG control [21,28] or H_∞ robust control [18,9]. Most of these works used a finite dimensional approximation of the system to design controllers. Recent approaches took into account the distributed feature of the system, either by using a semigroup approach [31,2], or by a Riemann invariants approach [12].

The methods developed using Riemann invariants provide a sufficient stability result for rectangular horizontal frictionless channels around a uniform flow regime. For more realistic cases, only vanishing perturbations can be considered [22]. This main limitation of the Riemann invariants method leads to consider an alternative method based on frequency domain approach. Such a method is very close to the one classically used by control engineers: the nonlinear PDE is first linearized around a stationary regime, then the Laplace transform is used to consider the linearized PDE in the frequency domain, and classical frequency domain tools are used to design controllers, in a very similar way as when the system is represented by finite dimensional transfer functions.

The objective of this paper is to consider this approach with a rigorous perspective, and to show what can effectively be guaranteed by using such a frequency domain approach for hyperbolic conservation laws.

We have already developed such an approach in previous papers [18,19], by considering stability with respect to input perturbations. Here, we also consider the stabilization of the system for non zero initial conditions.

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The main results of the paper are as follows:

(1) We provide a detailed characterization of the transfer matrix of the considered hyperbolic system, and show that it belongs to the class $\hat{\mathcal{B}}(\sigma)$ of Callier-Desoer [6],
(2) We use Nyquist theorem to derive a necessary and sufficient condition for input-output stability of the boundary controlled hyperbolic system,
(3) We clarify the link between input-output and internal stability.

We also examine in detail the specific case of diagonal boundary control and extend the results presented by [12].

These results are illustrated for boundary control of linearized Saint-Venant equations, representing open channel flow around a given stationary regime.

2 CONTROL PROBLEM STATEMENT AND EXISTENCE OF SOLUTIONS

2.1 Control problem

We consider the following linear system of hyperbolic conservation laws:

$$\frac{\partial \xi}{\partial t} + \begin{pmatrix} 0 & 1 \\ \alpha\beta & \alpha-\beta \end{pmatrix} \frac{\partial \xi}{\partial x} + \begin{pmatrix} 0 & 0 \\ -\gamma & \delta \end{pmatrix} \xi = 0 \qquad (1)$$

where t and x are the two independent variables : a time variable $t \in [0, +\infty)$ and a space variable $x \in [0, L]$ on a finite interval, $\xi(x,t) = (h(x,t), q(x,t))^T : [0, L] \times [0, +\infty) \to \Omega \in \mathbb{R}^2$ is the state of the system. $\alpha > \beta > 0$, $\gamma \geq 0$ and $\delta \geq 0$ are positive real constants.

The first equation of system (1) can be interpreted as a mass conservation law with h the conserved quantity and q the flux. The second equation can then be interpreted as a momentum conservation law.

We consider the solutions of the Cauchy problem for the system (1) over $[0, L] \times [0, +\infty)$ under an initial condition $\xi(0, x) = \xi_0(x)$, $x \in [0, L]$ and two boundary conditions of the form $q(0,t) = q_0(t)$ and $q(L,t) = q_L(t)$, $t \in [0, +\infty)$.

2.2 Existence and well-posedness

Following a classical approach, we introduce the bounded group $\mathbb{T}(t)$ on $\mathcal{L}_2([0, L], \mathbb{R}^2)$, generated by the following linear operator:

$$A_1 \xi = \begin{pmatrix} 0 & 1 \\ \alpha\beta & \alpha-\beta \end{pmatrix} \frac{\partial \xi}{\partial x} + \begin{pmatrix} 0 & 0 \\ -\gamma & \delta \end{pmatrix} \xi \qquad (2)$$

where A_1 is then defined on the domain in $\mathcal{L}_2([0, L], \mathbb{R}^2)$ consisting of functions $\xi \in H^1([0, L], \mathbb{R}^2)$ that vanish at $x = 0$ and $x = L$. $H^1([0, L], \mathbb{R}^2)$ corresponds to the Sobolev space of $\mathbb{R}^2$ functions whose derivatives (in generalized sense) are square integrable on $[0, L]$.

2.2.1 Continuous solutions

The study of the properties of solutions of linear hyperbolic partial differential equations is a classical problem that has been deeply investigated in many references (see e.g. [24,25] and references therein). In the sequel, we only recall some basic facts, the arguments and proofs can thus be found in the cited references.

First of all, the existence and uniqueness of the solution can be proved using the characteristics system, which enables to restate the PDE as a set of classical ODEs (see the discussion preceding Theorem 2.1. in [24]). Then, if $\xi_0(x)$ and $u(t) = (q_0(t), q_L(t))^T$ are two continuously differentiable functions of their argument, one can show that the solutions of system (1) are continuously differentiable with respect to their arguments, *i.e.*, $\xi(x,t) \in \mathcal{C}^1([0, L] \times [0, \infty), \mathbb{R}^2)$. Furthermore, based on a slight extension of Theorem 2.1 in [24], there exist two finite constants $M > 0$ and η such that for any $t \in [0, \infty)$, any $\xi \in \mathcal{C}^1([0, L], \mathbb{R}^2)$ and any $u^{(t)} \in \mathcal{L}_2([0, t], \mathbb{R}^2) \cap \mathcal{C}^1([0, t], \mathbb{R}^2)$, there exists a finite constant K_t such that

$$\|\xi(\cdot, t)\|_{\mathcal{L}_2([0,L],\mathbb{R}^2)} \leq M e^{\eta t} \|\xi_0\|_{\mathcal{L}_2([0,L],\mathbb{R}^2)} + K_t \|u^{(t)}\|_2, \qquad (3)$$

where $u^{(t)}$ denotes the restriction of u to $[0, t]$.

2.2.2 Generalized solutions

Following this preliminary result and the fact that the continuous differentiable functions defined on any finite support are dense in $\mathcal{L}_2$, it is then possible to handle the inputs and the initial conditions in $\mathcal{L}_2([0, t], \mathbb{R}^2)$ and $\mathcal{L}_2([0, L], \mathbb{R}^2)$ respectively.

We thus conclude that for any $t \in [0, \infty)$, any $(q_0, q_L) \in \mathcal{L}_2([0, t], \mathbb{R}^2)$ and any $\xi_0 \in \mathcal{L}_2([0, L], \mathbb{R}^2)$ there exists a unique generalized solution belonging to $\mathcal{C}([0, t], \mathcal{L}_2([0, L], \mathbb{R}^2))$.

Furthermore, the solution of system (1) can be rewritten as

$$\xi(\cdot, t) = \Phi(t) u^{(t)} + \mathbb{T}(t) \xi_0$$

where $\Phi(t)$ is a bounded linear operator defined from $\mathcal{L}_2([0, t], \mathbb{R}^2)$ into $\mathcal{L}_2([0, L], \mathbb{R}^2)$. Finally, Theorem 3.1 in [25] guarantees that the generalized solution also satisfies inequality (3).

It remains to ensure that the output of the system is well-defined, *i.e.*, for any $t \in [0, \infty)$, any $\xi_0 \in \mathcal{L}_2([0, L], \mathbb{R}^2)$ and any $(q_0, q_L) \in \mathcal{L}_2([0, t], \mathbb{R}^2)$, $y(t) = (h(0,t), h(L,t))$

belongs to $\mathcal{L}_2([0,t],\mathbb{R}^2)$. As in the case of the existence of generalized solutions, the main idea in this context is to use density type argument. We do not develop the details of the proof since it can be easily adapted from the one associated to example 4.3.12 in [11].

3 FREQUENCY DOMAIN ANALYSIS

3.1 Open-loop transfer matrix

Using the above results, we know that the solutions of (1) are Laplace transformable, which enables us to use a frequency domain approach. The system's open-loop transfer matrix can then be obtained by applying Laplace transform to the linear partial differential equations (1), and solving the resulting system of Ordinary Differential Equations in the variable x, parameterized by the Laplace variable s [16]. In this case, using the classical relation $\frac{\hat{df}}{dt} = s\hat{f}(s) - f(0)$ and after elementary manipulations, we get:

$$\frac{\partial\hat{\xi}(x,s)}{\partial x} = A(s)\hat{\xi}(x,s) + B\xi(x,0) \tag{4}$$

with

$$A(s) = \frac{1}{\alpha\beta}\begin{pmatrix}(\alpha-\beta)s+\gamma & -s-\delta \\ -\alpha\beta s & 0\end{pmatrix}$$
$$B = \frac{1}{\alpha\beta}\begin{pmatrix}(\beta-\alpha) & 1 \\ \alpha\beta & 0\end{pmatrix}.$$

The general solution of (4) is then given by:

$$\hat{\xi}(x,s) = e^{A(s)x}\left[\hat{\xi}(0,s) + \bar{\xi}_0(x,s)\right] \tag{5}$$

with $\bar{\xi}_0(x,s) = \int_0^x e^{-A(s)v}B\xi(v,0)dv$.

The state $\hat{\xi}(x,s)$ is then obtained with the transition matrix $\Gamma(x,s) = e^{A(s)x}$ acting on the sum of two terms: the first one $\hat{\xi}(0,s)$ is the boundary condition in $x = 0$, and the second one $\bar{\xi}_0(x,s)$ is linked to the initial condition at $t = 0$.

The Laplace transform also enables to derive from Eq. (1) the distributed transfer matrix expressing the state of the system $\hat{\xi}(x,s) = (\hat{h}(x,s),\hat{q}(x,s))^T$ at each point $x \in [0,L]$ of the system as a function of the boundary inputs $\hat{u}(s) = (\hat{q}(0,s),\hat{q}(L,s))^T$ and initial conditions:

$$\hat{\xi}(x,s) = G(x,s)\hat{u}(s) + G_0(x,s)\bar{\xi}_0(L,s) + \Gamma(x,s)\bar{\xi}_0(x,s) \tag{6}$$

with $G_0(x,s) = G(x,s)\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix} - \Gamma(x,s)$, $\Gamma(x,s) = e^{A(s)x}$ and $G(x,s) = (g_{ij}(x,s))$, where

$$g_{11}(x,s) = \frac{\lambda_2 e^{\lambda_2 x+\lambda_1 L} - \lambda_1 e^{\lambda_1 x+\lambda_2 L}}{s(e^{\lambda_2 L} - e^{\lambda_1 L})} \tag{7}$$
$$g_{12}(x,s) = \frac{\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}}{s(e^{\lambda_2 L} - e^{\lambda_1 L})} \tag{8}$$
$$g_{21}(x,s) = \frac{e^{\lambda_1 x+\lambda_2 L} - e^{\lambda_2 x+\lambda_1 L}}{e^{\lambda_2 L} - e^{\lambda_1 L}} \tag{9}$$
$$g_{22}(x,s) = \frac{e^{\lambda_2 x} - e^{\lambda_1 x}}{e^{\lambda_2 L} - e^{\lambda_1 L}} \tag{10}$$

λ_1 and λ_2 are the eigenvalues of $A(s)$, given by, for $i = 1, 2$:

$$\lambda_i(s) = \frac{(\alpha-\beta)s + \gamma + (-1)^i\sqrt{d(s)}}{2\alpha\beta} \tag{11}$$

with $d(s) = (\alpha+\beta)^2s^2 + 2[(\alpha-\beta)\gamma + 2\alpha\beta\delta]s + \gamma^2$. Dependence in s is omitted in equations (7–10) for simplicity.

Specifying the outputs $\hat{y}(s) = (\hat{h}(0,s),\hat{h}(L,s))^T$, we get the following representation:

$$\hat{y}(s) = P(s)\hat{u}(s) + P_0(s)\bar{\xi}_0(L,s) \tag{12}$$

where $P_0(s) = P(s)\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix} - \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}$, and $P(s) = (p_{ij}(s))$, with

$$p_{11}(s) = g_{11}(0,s) \tag{13}$$
$$p_{12}(s) = g_{12}(0,s) \tag{14}$$
$$p_{21}(s) = g_{11}(L,s) \tag{15}$$
$$p_{22}(s) = g_{12}(L,s) \tag{16}$$

3.1.1 Open-loop poles of the system

The poles of this transfer matrix are obtained as the solutions of

$$s(e^{\lambda_2(s)L} - e^{\lambda_1(s)L}) = 0.$$

There is a pole in zero (the hyperbolic system acts as an integrator for the variable $h(x,t)$ with the considered boundary conditions) and the other poles verify the following equation:

$$d(s) = -\frac{4\alpha^2\beta^2k^2\pi^2}{L^2}$$

with $k \in \mathbb{N}^*$.

The poles $(p_{\pm k})_{k\in\mathbb{N}^*}$ are then given by:

$$p_{\pm k} = \frac{2\alpha\beta}{(\alpha+\beta)^2}\left[-\delta - \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\frac{\gamma}{2} \pm \sqrt{\Delta(k)}\right] \quad (17)$$

with $\Delta(k) = \delta^2 - \frac{\gamma^2}{\alpha\beta} + \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\gamma\delta - \frac{k^2\pi^2(\alpha+\beta)^2}{L^2}$.

Let $k_m \in \mathbb{N}^*$ be the greatest integer such that $\Delta(k_m) \geq 0$. Then the poles obtained for $0 < k \leq k_m$ are negative real, and those obtained for $k > k_m$ are complex conjugate, with a constant real part equal to $-\frac{2\alpha\beta}{(\alpha+\beta)^2}\left[\delta + \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\frac{\gamma}{2}\right]$. The oscillating poles are therefore located on a vertical line in the left half plane. Let us note that when $\gamma = \delta = 0$ the poles are located on the imaginary axis. For simplicity, we assume in the following that the poles have single multiplicity, i.e. that $\Delta(k) \neq 0$.

3.1.2 Properties of the transfer matrix

We show in the sequel that the transfer matrix of system (1) belongs to the Callier-Desoer algebra [6,7]. The fact that the system belongs to the Callier-Desoer algebra is of great interest in the control context. Typically, that allows to ensure that the closed-loop system is well-defined and leads to necessary and sufficient conditions for the internal stability of the closed-loop system. Furthermore, the stability conditions can be tested with the help of the famous Nyquist criteria (see [6]).

Let $\sigma \in \mathbb{R}$ be a given real number, and $\mathcal{A}(\sigma)$ denote the set of distributions f such that:

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sum_{i=0}^{\infty} f_i\delta(t-t_i) + f_a(t) & \text{if } t \geq 0, \end{cases}$$

where $f_a(t)e^{-\sigma t} \in L_1(0,\infty)$, $\delta(.)$ represents the unit delta distribution, $0 \leq t_0 < t_1 < \ldots$ and f_i are real constants, and $\sum_{i=0}^{\infty} |f_i|e^{-\sigma t_i} < \infty$.

$\hat{\mathcal{A}}(\sigma)$ denotes the set of all functions $\hat{f} : \mathbb{C}^+ \to \mathbb{C}$ that are Laplace transforms of elements of $\mathcal{A}(\sigma)$; they are analytic and bounded in $\Re(s) \geq \sigma$, where $\Re(s)$ denotes the real part of s.

The sets $\mathcal{A}_-(\sigma)$ and $\hat{\mathcal{A}}_-(\sigma)$ are defined by:

$$\mathcal{A}_-(\sigma) = \bigcup_{\sigma_1<\sigma} \mathcal{A}(\sigma_1) \text{ and } \hat{\mathcal{A}}_-(\sigma) = \bigcup_{\sigma_1<\sigma} \hat{\mathcal{A}}(\sigma_1)$$

$\hat{\mathcal{A}}_-^\infty(\sigma)$ denotes the set of elements $\hat{b} \in \hat{\mathcal{A}}_-(\sigma)$ being bounded away from zero at infinity in $\Re(s) \geq \sigma$.

The set $\hat{\mathcal{B}}(\sigma)$ consists of all functions $\hat{f} = \hat{a}/\hat{b}$, where $\hat{a} \in \hat{\mathcal{A}}_-(\sigma)$ and $\hat{b} \in \hat{\mathcal{A}}_-^\infty(\sigma)$. $\hat{\mathcal{B}}(\sigma)$ is an algebra, as shown by [6,8].

Using the above definitions, we state the following proposition.

Proposition 1 *Each element $p_{ij}(s)$ of the transfer matrix $P(s)$ belongs to the Callier-Desoer algebra $\hat{\mathcal{B}}(\sigma)$ if and only if $\gamma > 0$ or $\delta > 0$, with $\sigma > -\frac{2\alpha\beta}{(\alpha+\beta)^2}\left[\delta + \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\frac{\gamma}{2}\right]$.*

Proof Using the closed-form expression of the poles of $P(s)$, $p_{ij}(s)$ can be decomposed as an infinite sum (see proof in appendix):

$$p_{ij}(s) = c_{ij} + \frac{a_{ij}^{(0)}}{s} + \sum_{k=-\infty,k\neq0}^{\infty} \frac{a_{ij}^{(k)}s}{p_k(s-p_k)}$$

with c_{ij} and $a_{ij}^{(k)}$ constant scalars, defined by:

$$a_{ij}^{(k)} = \lim_{s\to p_k}(s-p_k)p_{ij}(s) \quad (18)$$

and

$$c_{ij} = \frac{d}{ds}[sp_{ij}(s)]|_{s=0}. \quad (19)$$

Then, $p_{ij}(s)$ is the sum of an unstable finite dimensional part and a stable infinite dimensional part belonging to $\hat{\mathcal{A}}_-(\sigma)$. Since the stable infinite dimensional part has finitely many poles with real part larger than $\sigma_1 = -\frac{2\alpha\beta}{(\alpha+\beta)^2}\left[\delta + \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\frac{\gamma}{2}\right]$, then $p_{ij}(s) \in \hat{\mathcal{B}}(\sigma)$, with $\sigma > \sigma_1$ (see theorem 3 in [8]).

Finally, $P(s) \in M(\hat{\mathcal{B}}(\sigma))$, which is the multivariable extension of $\hat{\mathcal{B}}(\sigma)$. ■

Remark 1 *If $\gamma = \delta = 0$, the open-loop poles of the system are located on the imaginary axis, therefore the system has an infinite number of marginally stable poles, and does not belong to $\hat{\mathcal{B}}(0)$ [11].*

3.2 Closed-loop transfer matrix

Let $K(s)$ denote the Laplace transform of the finite dimensional controller K, i.e.:

$$\hat{u}(s) = K(s)\hat{y}(s) + \hat{p}(s) \quad (20)$$

with $K(s) = \begin{pmatrix} k_{11}(s) & k_{12}(s) \\ k_{21}(s) & k_{22}(s) \end{pmatrix}$ and where $\hat{p} = (\hat{p}_1\ \hat{p}_2)^T$ is the Laplace transform of the input perturbation.

Then the control input u is given by:

$$\hat{u}(s) = S_u\hat{p}(s) + S_uKP_0\bar{\xi}_0(L,s) \tag{21}$$

with $S_u = (I - KP)^{-1}$ the input sensitivity function and the outputs by:

$$\hat{y}(s) = PS_u\hat{p}(s) + S_yP_0\bar{\xi}_0(L,s) \tag{22}$$

with $S_y = (I - PK)^{-1}$ the output sensitivity function.

If the controller $K(s)$ belongs to $M\hat{\mathcal{B}}(\sigma)$, then the feedback interconnection also belongs to $M\hat{\mathcal{B}}(\sigma)$ provided $\det(I - K(s)P(s))$ is bounded away from zero at infinity in $\mathbb{C}_{\sigma^+}$. This condition is not so easy to check in practice. Nevertheless, when the open-loop system is strictly proper, this last condition could be easily tested by Nyquist criteria, indeed, we have the following theorem (from theorem 9.1.8 page 463 of [11]):

Theorem 1 *Suppose that $K(s) \in M\hat{\mathcal{B}}(0)$ has p_K poles in $\overline{\mathbb{C}^+}$, counted according to their McMillan degree. Then, K is a stabilizing controller for P if and only if:*

$$ind(\det(I - KP)) = -1 - p_K$$

where $ind(\det(I - KP))$ denotes the Nyquist index of $\det(I - KP)$, i.e. the number of times the plot of $\det(I - K(s)P(s))$ encircles the origin in a counter-clockwise sense as s decreases from $\jmath\infty$ to $-\jmath\infty$ on the indented imaginary axis.

In that case, the Nyquist criteria is easy to use and its application is close to the one use classically for strictly proper rational transfer functions (see e.g. [11]).

When the open-loop is not strictly proper, we have to take care of the behavior of Nyquist graph at the infinity. Following [4,5], it is still possible to use Nyquist theorem in this case, but additional conditions have to be checked. This case will be considered in section 5, due to its practical importance for hyperbolic systems where boundary conditions are imposed by physical constraints, corresponding to non strictly proper diagonal controllers.

3.3 From input to state

We now generalize the transfer function approach by using the *distributed* transfer function, which relates the inputs to the state $\hat{\xi}(x,s)$.

Using equations (6) and (21), the distributed closed-loop transfer matrix is written as:

$$\begin{aligned}\hat{\xi}(x,s) = {} & G(x,s)S_u\hat{p}(s) + G(x,s)S_uN_0\bar{\xi}_0(L,s)\\ & +\Gamma(x,s)\left[\bar{\xi}_0(x,s) - \bar{\xi}_0(L,s)\right]\end{aligned}$$

with $N_0(s) = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} - K\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}$.

The poles of the closed-loop distributed transfer matrix $G(x,s)S_u(s)N_0(s)$ are identical to the ones of the closed-loop input-output transfer matrix $P(s)S_u(s)$, only the zeros change. This is due to the fact that the feedback is applied only at the boundaries. Therefore, the results obtained in the last section for the external stability can be directly generalized to the state, since for any x the transfer $G(x,s)S_u(s)N_0(s)$ belongs to H_∞.

The Nyquist theorem enables to extend classical results for finite dimensional systems to infinite dimensional systems belonging to the Callier-Desoer algebra. However, it only provides an input-output or external stability result. In the next section, we provide Lyapunov type stability result.

4 LYAPUNOV STABILITY ANALYSIS

We show in this section that the input-output stability of the closed-loop system implies the exponential stability of the state of the system.

We now recall a result allowing to strongly relate input-output stability and Lyapunov stability for systems possessing a minimal state-space realization. This result finds its roots in the dissipativity framework introduced by Willems in his seminal paper [30]. It should be noted that Willems mentioned in [29] that his work extended to dynamical systems a previous result of Baker and Bergen [1] in the context of linear infinite dimensional systems.

In the sequel, Σ is a causal linear time-invariant system such that for any input u in $\mathcal{L}_2([0,t],\mathbb{R}^p)$, its output given by $y = \Sigma(u)$ belongs to $\mathcal{L}_2([0,t],\mathbb{R}^m)$ (Σ is thus assumed well-defined). Z is a normed vectorial space equipped with the norm $\|\cdot\|_Z$ and corresponds to the the state-space of Σ. Finally, the state of Σ at time $t \in [0,\infty)$ belonging to Z is denoted by $z(t)$ and it is formally related to the input u and the initial condition by the following causal relation: $z(t) = \phi(t, 0, z(0), u^{(t)})$.

The following definition corresponds to the uniform reachability and the uniform observability defined by Willems in [29] for causal linear invariant systems.

Definition 1 *Σ is said to be minimal if:*

- *it is uniformly reachable from $z(0) = 0$, i.e., there exist $\alpha_r > 0$ and $T_r > 0$ such that for any $z \in Z$ there exists $u_r \in \mathcal{L}_2([0,T_r],\mathbb{R}^p)$ such that $z(0) = 0$, $z = z(T_r) = \phi(T_r, 0, 0, u_r^{(T_r)})$ and $\int_0^{T_r} \|u_r(\tau)\|^2 d\tau \leq \alpha_r^2\|z\|_Z^2$ and*

- *it is uniformly observable, i.e., there exist $\beta_o > 0$ and $T_o > 0$ such that for any $z \in Z$ and $u = 0$, we have $\int_0^{T_o} \|y(\tau)\|^2 d\tau \geq \beta_o^2 \|z\|_Z^2$.*

Proposition 2 *Let Σ be a causal linear time invariant system defined from $\mathcal{L}_2([0,t],\mathbb{R}^p)$ into $\mathcal{L}_2([0,t],\mathbb{R}^m)$. If Σ is finite gain stable on $\mathcal{L}_2$, i.e if there exists $\eta \geq 0$ such that $\|y\|_2 \leq \eta\|u\|_2$ for any $u \in \mathcal{L}_2([0,\infty),\mathbb{R}^p)$ and if its state-space realization is minimal then Σ is uniformly exponentially stable, i.e. there exist a and b positive such that for any $z(0) \in Z$, we have $\|z(t)\|_Z \leq ae^{-bt}\|z(0)\|_Z$ for any $t \geq 0$.*

Proof See appendix. ■

Actually if the closed-loop system is internally stable then the map between (d_1, d_2) to (y_1, y_2) is $\mathcal{L}_2$ gain stable (since the closed-loop matrix belongs to H_∞) and thus only the minimality of the state-space realization of the closed-loop operator has to be proved.

In our context, the state-space of the closed-loop system is given by the concatenation of the state-space of the hyperbolic system given by (1) and the one of the controller K. We then deduce that $Z = \mathcal{L}_2([0,L],\mathbb{R}^2)$ and $z = \xi$ when a constant feedback is considered. When K is a finite dimensional time-invariant linear controller of order n, then $Z = \mathcal{L}_2([0,L],\mathbb{R}^2) \times \mathbb{R}^n$ with $z = (\xi, x_K)$ where x_K is the state of K. In this last case, Z is equipped with the following norm:

$$\|z\|_Z = \left(\|\xi\|^2_{\mathcal{L}_2([0,L],\mathbb{R}^2)} + \|x_K\|^2\right)^{1/2}.$$

Corollary 1 *Let $K(s)$ a finite dimensional controller with a minimal realization $K = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. If the closed-loop is stable then the closed-loop system is uniformly exponentially stable.*

Proof If we assume that the state-space realization of the controller is such that (A, B) is controllable and (A, C) is observable, it is straightforward to prove that the state-space realization of K is minimal following definition 1 [3].

Based on the results presented in [23] and [25], it is then possible to prove that the minimality of closed-loop system holds if the hyperbolic system given by (1) is also minimal.

Actually, the state-space of system (1) is reachable from $\xi_0 = 0$, i.e. there exist two finite constants $T_r > 0$ and $\alpha_r > 0$ such that for any $\xi_1 \in \mathcal{L}_2([0,L],\mathbb{R}^2))$ there exists $u \in \mathcal{L}_2([0,T],\mathbb{R}^2)$ such that $\xi_1(\cdot,T_r) = \Phi(T_r)u^{(T_r)}$ and with

$$\|u\|_{\mathcal{L}_2([0,T_r],\mathbb{R}^2)} \leq \alpha_r\|\xi_1(\cdot,T_r)\|_{\mathcal{L}_2([0,L],\mathbb{R}^2)}.$$

Using the duality between controllability and observability (see *e.g.* [25]), it is also possible to prove that system (1) is observable, *i.e.*, there exist two finite constants $T_o > 0$ and $\beta_o > 0$ such that for any $\xi_1 \in \mathcal{L}([0,L],\mathbb{R}^2))$, we have

$$\|y\|_{\mathcal{L}_2([0,T_o],\mathbb{R}^2)} \geq \beta_o\|\xi_0\|_{\mathcal{L}_2([0,L],\mathbb{R}^2)}$$

where y corresponds to the output of system (1) initialized at $\xi(\cdot,0) = \xi_0$ and where $u(t) = 0$ for $t \in [0,T_o]$. ■

5 SPECIFIC CASE OF STATIC DIAGONAL BOUNDARY CONTROL

Static proportional diagonal controllers are commonly encountered (gates in the case of open channels lead to static boundary control), and have been studied in the literature (see e.g. [12]). In this case, the closed-loop system simplifies.

In the general case, we use a classical result providing a necessary and sufficient condition for the invertibility of a operator in $\mathcal{A}$. This result then provides a necessary and sufficient condition for the closed-loop system internal stability (see [13]). One of the interest of the given conditions is the possibility to check it using an extended version of the classical Nyquist graphical test, even if as already pointed out, we have to take care of the behavior of the Nyquist plot at the infinity since the open-loop system is not strictly proper.

In the case $\gamma = \delta = 0$, we study the poles of the closed-loop system, and derive an analytical necessary and sufficient condition for the closed-loop poles damping to be larger than μ for proportional diagonal boundary control.

We now consider a static diagonal dynamic boundary controller defined by:

$$K = \begin{pmatrix} k_0 & 0 \\ 0 & k_L \end{pmatrix} \tag{23}$$

where k_0, k_L are constant scalars and we want to determine conditions on (k_0, k_L) such that the closed-loop system is stable.

5.1 General case

Let us first study the general case, where $\gamma \neq 0$ or $\delta \neq 0$. Following the remarks done in section 2.2, since the transfer matrix belongs to the Callier-Desoer algebra, we already know that the closed-loop system is then well-defined. We moreover have this necessary and sufficient condition for the closed loop stability (see theorem 36 page 90 in [13]):

Theorem 2 *The closed-loop system is stable if and only if*

$$(i) \qquad \inf_{\Re(s)>0} |\det(I - KP(s))| > 0 \tag{24}$$

$$(ii) \det(D(0) - KN(0) - KP_b(0)D(0)) \neq 0 \tag{25}$$

where P_u is the unstable part of P, $P_b = P - P_u$, and $(N(s), D(s))$ is a right coprime factorization of $P_u(s)$.

The condition (i) of theorem 2 is actually the basis of the famous Nyquist criteria allowing to test condition (i) through the examen of the behavior of the determinant map for s covering only the imaginary axis. In our case, the open-loop is non strictly proper, and the application of the Nyquist criteria is more delicate.

The second condition can be simplified using the expression of the coprime factors of $P_u(s)$:

$$P_u(s) = \frac{1}{s}\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} \end{pmatrix}$$

with

$$a_{11}^{(0)} = \lim_{s\to 0} sp_{11}(s) = \frac{\gamma}{\alpha\beta(e^{\frac{\gamma L}{\alpha\beta}} - 1)} = -a_{12}^{(0)} \tag{26}$$

$$a_{21}^{(0)} = \lim_{s\to 0} sp_{21}(s) = \frac{\gamma}{\alpha\beta(1 - e^{\frac{-\gamma L}{\alpha\beta}})} = -a_{22}^{(0)} \tag{27}$$

A coprime factorization of P_u is expressed as:

$$P_u(s) = N(s)D(s)^{-1} \tag{28}$$

with $N(s) = \frac{1}{s-(n_1-n_2)}\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} \end{pmatrix}$ and $D(s) = \frac{1}{s-(n_1-n_2)}\begin{pmatrix} s+n_2 & -n_1 \\ n_2 & s-n_1 \end{pmatrix}$, where n_1 and n_2 are constant scalars such that $n_1 < n_2$.

One can also directly compute $P_b(0)$, since we have:

$$P_b(0) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

with c_{ij} given by (19).

Finally, using the expressions of c_{ij} given in appendix and the coprime factorization (28), condition (ii) reduces to:

$$\frac{k_0(1 - k_L c_2)}{k_L(1 - k_0 c_1)} \neq e^{\psi}$$

with $c_1 = \frac{\delta}{\gamma}\frac{e^{\psi}-1-\psi}{e^{\psi}-1}$, $c_2 = \frac{\delta}{\gamma}\frac{(1-\psi)e^{\psi}-1}{e^{\psi}-1}$ and $\psi = \frac{\gamma L}{\alpha\beta}$.

This condition can easily be tested numerically, while the first condition of theorem 2 is more difficult to test in practice. This difficulty is only due to the fact that the controller is not strictly proper. We propose below a way to circumvent this problem by using an asymptotic analysis for high frequencies. Let us first provide a necessary condition of stability.

Proposition 3 *The following inequality is a necessary condition of stability:*

$$\left|\frac{(\beta + k_0)(\alpha - k_L)}{(\alpha - k_0)(\beta + k_L)}\right| < e^{(r_1+r_2)L} \tag{29}$$

with $r_1 = \frac{\alpha\delta-\gamma}{\alpha(\alpha+\beta)}$ and $r_2 = \frac{\beta\delta+\gamma}{\beta(\alpha+\beta)}$.

Proof For $|s| \gg \frac{2[(\alpha-\beta)\gamma+2\alpha\beta\delta]}{(\alpha+\beta)^2}$, the eigenvalues can be approximated by:

$$\lambda_1(s) = -r_1 - \frac{s}{\alpha} + O(1/s) \tag{30}$$

$$\lambda_2(s) = r_2 + \frac{s}{\beta} + O(1/s). \tag{31}$$

Then, using a continuity argument, one may show that if inequality (29) is not verified, there exists R such that the closed-loop poles with modulus larger than R are unstable. Therefore the condition (29) is a necessary condition for stability. ∎

Now, using this property, we can restrict the domain where condition (24) needs to be tested. This is stated in the following corollary.

Corollary 2 *If condition (29) is verified, then there exists $R_0 > 0$ such that condition (i) of theorem 2 needs only be tested on a finite range $|s| < R_0$.*

Proof See appendix. ∎

Therefore, one may use the classical Nyquist graphical criterion to test condition i) of theorem 2 on a finite range of frequencies.

To summarize, we have obtained a necessary and sufficient condition of stability that can be tested using classical methods such as the Nyquist plot for finite dimensional systems, and two algebraic conditions that can easily be tested numerically.

5.2 Case $\delta = \gamma = 0$

We now consider the special case where $\delta = \gamma = 0$, which corresponds to the system considered by several authors (see e.g. [12]). In this case, the transfer matrix no longer belongs to the class $\hat{\mathcal{B}}(0)$ and can only be stabilized by a non strictly proper controller [11]. Therefore, the Nyquist criterion does not apply. It can nevertheless be shown that it belongs to the class of regular transfer functions and then well-posedness of the closed-loop can be guaranteed (see [26,2], references therein and the result of section 2.2).

Moreover, a necessary and sufficient condition can be derived from the closed-form expression for the poles of the closed-loop system.

Proposition 4 *Let $\mu \geq 0$ be a positive real number. The closed-loop poles $p_k, k \in \mathbb{Z}$ verify $\Re(p_k) < -\mu$ if and only if the couple (k_0, k_L) verifies the following inequality:*

$$\left| \frac{(\beta + k_0)(\alpha - k_L)}{(\alpha - k_0)(\beta + k_L)} \right| < e^{-\mu\tau} \tag{32}$$

with $\tau = L\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$.

Proof In this case, the eigenvalues are given by $\lambda_1(s) = -\frac{s}{\alpha}$ and $\lambda_2(s) = \frac{s}{\beta}$. Then, the poles are solutions of:

$$e^{\tau s} = \frac{(\beta + k_0)(\alpha - k_L)}{(\alpha - k_0)(\beta + k_L)} \tag{33}$$

The closed-loop poles are then given by:

$$p_k = \frac{1}{\tau} \log\left(\frac{(\beta + k_0)(\alpha - k_L)}{(\alpha - k_0)(\beta + k_L)} \right) + \frac{2\jmath k\pi}{\tau}$$

where the complex form of the logarithm is used. The property derives directly from the poles expression. ■

This condition extends the sufficient condition obtained by [12], as shown below in section 6.

Let us now examine the implications of (32) for specific values of (k_0, k_L).

When $k_0 = 0$, i.e. for simple boundary control at $x = L$, and for $\mu = 0$, the condition (32) reduces to:

$$\left| \frac{1 - k_L/\alpha}{1 + k_L/\beta} \right| < 1.$$

Since the function $k_L \mapsto \left| \frac{1-k_L/\alpha}{1+k_L/\beta} \right|$ is always lower than 1 for any $k_L > 0$, this condition is always satisfied. Therefore, any positive proportional boundary controller at $x = L$ stabilizes the system (1). When $k_L = \alpha$, the left hand side is zero. This corresponds to the optimal gain for damping of oscillating modes (see [19]).

When $k_L = 0$, i.e. for simple upstream boundary control, and for $\mu = 0$, the condition reduces to:

$$\left| \frac{1 + k_0/\beta}{1 - k_0/\alpha} \right| < 1.$$

In this case, the function $k_0 \mapsto \left| \frac{1+k_0/\beta}{1-k_0/\alpha} \right|$ is lower than 1 for $-\frac{2\alpha\beta}{\alpha-\beta} < k_0 < 0$. Therefore, contrarily to the boundary control case at $x = L$, the closed-loop system with boundary control at $x = 0$ is not stable for any $k_0 < 0$. When $k_0 = -\beta$, the left hand side is zero. This also corresponds to the optimal gain for damping of oscillating modes in the case of boundary control at $x = 0$.

6 APPLICATION TO BOUNDARY CONTROL OF AN OPEN-CHANNEL

6.1 Saint-Venant equations

We apply the result of the paper to the control of a prismatic canal pool of length L with uniform geometry (not necessarily rectangular) and a given slope $S_b \geq 0$, represented by the Saint-Venant equations involving the average discharge $Q(x,t)$ and the water depth $H(x,t)$ along one space dimension [10]:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{34}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial Q^2/A}{\partial x} + gA\frac{\partial H}{\partial x} = gA\left(S_b - \frac{Q^2 n^2}{A^2 R^{4/3}}\right) \tag{35}$$

where $A(x,t)$ is the wetted area (m^2), $Q(x,t)$ the discharge (m^3/s) across section A, $V(x,t)$ the average velocity (m/s) in section A, $H(x,t)$ the water depth (m), g the gravitational acceleration (m/s^2), n the Manning coefficient (sm$^{-1/3}$) and R the hydraulic radius (m), defined by $R = A/P$, where P is the wetted perimeter (m).

The boundary conditions are $Q(0,t) = Q_0(t)$ and $Q(L,t) = Q_L(t)$, and the initial conditions are given by $Q(x,0)$ and $H(x,0)$.

6.2 Linearized Saint-Venant equations

We consider small variations of discharge $q(x,t)$ and water depth $h(x,t)$ around constant stationary values Q_0 (m^3/s) and H_0 (m). When $S_b \neq 0$, the equilibrium regime (H_0, Q_0) verifies the following algebraic equation:

$$S_b = \frac{Q_0^2 n^2}{A_0^2 R_0^{4/3}} \tag{36}$$

If the slope S_b is zero and $n = 0$, then any couple (H_0, Q_0) can be chosen as an equilibrium solution, provided that the Froude number $F_0 = V_0/C_0$ remains strictly lower than 1. V_0 is the average velocity and $C_0 = \sqrt{gA_0/T_0}$ the wave celerity, with T_0 the water surface top width.

Linearizing the Saint-Venant equations around these stationary values leads to a linear hyperbolic system of partial differential equations (1) with the following values of the constant parameters:

$$
\begin{aligned}
\alpha &= C_0 + V_0 \\
\beta &= C_0 - V_0 \\
\gamma &= gS_b\left(\frac{10}{3} - \frac{4A_0}{3T_0P_0}\frac{dP_0}{dH}\right) \\
\delta &= \frac{2gS_b}{V_0}.
\end{aligned}
$$

Note that the variable h is scaled by a factor T_0, i.e. Eq. (1) applies in fact to $h^* = T_0 h$, which is denoted h with an abuse of notation.

For illustration purposes, we will focus on diagonal proportional control.

6.3 Proportional control

6.3.1 Case $\gamma = \delta = 0$

We explore the link between our result and the stability condition obtained by [12] in the case of a horizontal frictionless channel. In [12], the control is expressed as :

$$
\begin{aligned}
v(0,t) &= -2\alpha_0 c(0,t) \\
v(L,t) &= 2\alpha_L c(L,t),
\end{aligned}
$$

where v and c are deviations from equilibrium values of velocity V_0 and celerity C_0 and α_0, α_L are positive constants such that $0 < \alpha_0 < 1$ and $0 < \alpha_L < 1$.

Expressed in terms of our boundary conditions, since $v = \frac{q}{T_0H_0} - \frac{V_0}{T_0H_0}h$ and $c = \frac{C_0}{2T_0H_0}h$ in rectangular geometry, we get:

$$\alpha_0 = -\frac{1}{C_0}(k_0 - V_0) \tag{37}$$

$$\alpha_L = \frac{1}{C_0}(k_L - V_0), \tag{38}$$

where k_0 and k_L are the gains of the boundary controls $q(0,s) = k_0 h(0,s)$ and $q(L,s) = k_L h(L,s)$.

Using eqs. (37–38), it is easy to show that condition (32) is equivalent to:

$$\left(\frac{1-\alpha_0}{1+\alpha_0}\right)\left(\frac{1-\alpha_L}{1+\alpha_L}\right) < e^{-\mu\tau}.$$

For $\mu = 0$, i.e. only for stabilization, we recover the sufficient condition obtained by [12] based on a Riemann invariants approach. The Laplace transform approach provides here a necessary and sufficient condition for stability.

Fig. 1 depicts the condition (32) for the hyperbolic system described in the following section, enforcing $\delta = \gamma = 0$.

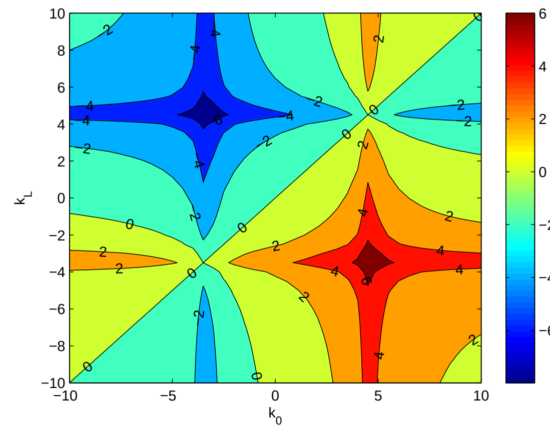

Fig. 1. Contour plot of function $(k_0, k_L) \mapsto \log\left(\frac{(\beta+k_0)(\alpha-k_L)}{(\alpha-k_0)(\beta+k_L)}\right)$.

This figure enables to select the control gains according to the desired damping for the closed-loop system in the case where $\delta = \gamma = 0$.

6.3.2 General case

The paper is illustrated for a canal pool of length $L = 3000$ m with a trapezoidal geometry, (bed width of 7 m, side slope of 1.5), a bed slope $S_b = 0.0001$ and Manning coefficient of 0.02. The considered stationary regime corresponds to a discharge $Q_0 = 14$ m^3/s and a water depth $H_0 = 2.12$ m. This leads to an hyperbolic system (1) with the following parameters $\alpha = 4.63$, $\beta = 3.33$, $\gamma = 2.7 \times 10^{-3}$, and $\delta = 3 \times 10^{-3}$.

Figures 2–3 depict the time domain simulation of static diagonal boundary controller for various values of (k_0, k_L). The initial state corresponds to a discharge deviation of 0.43 m^3/s from the equilibrium regime, and initial values of $h(0,0) = 0.509$ m and $h(0,L) = 0.536$ m. The hyperbolic system is simulated with a rational model of order 31 based on 15 pairs of poles.

When $(k_0, k_L) = (-3.33, 4.63)$, the two conditions of theorem 2 are fulfilled, and the closed-loop system is stable.

When $(k_0, k_L) = (0.5, 0.463)$, the sufficient condition provided by [12] is not fulfilled, since we have:

$$\left|\frac{(\beta+k_0)(\alpha-k_L)}{(\alpha-k_0)(\beta+k_L)}\right| = 1.1$$

However, the two conditions of theorem 2 are verified, which ensures that the closed-loop system is stable. However, it is clear that the damping is not as large as the one provided by the first case $(k_0, k_L) = (-3.33, 4.63)$. Let us note that these values correspond to the static gains which ensures the optimal damping of the oscillating modes. They correspond to the high frequency static gain of non-reflective upstream and downstream boundary controllers (see [19]).

Finally, when $(k_0, k_L) = (1, -2.8)$, the inequality (29) is not verified, therefore the closed-loop system is unstable, as can be checked in the simulation.

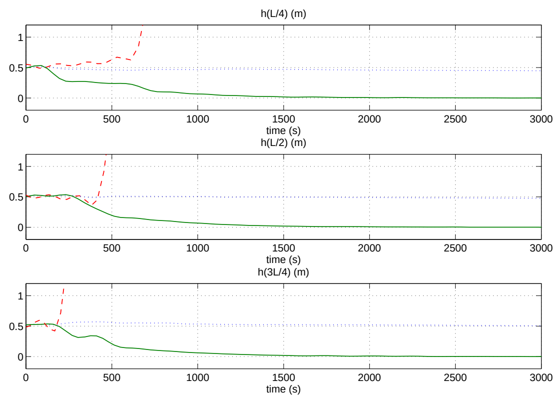

Fig. 2. Water level deviations along time for various values of (k_0, k_L): $(-3.33, 4.63)$ (solid line), $(0.5, 0.463)$ (dotted line), $(1, -2.8)$ (dashed line)

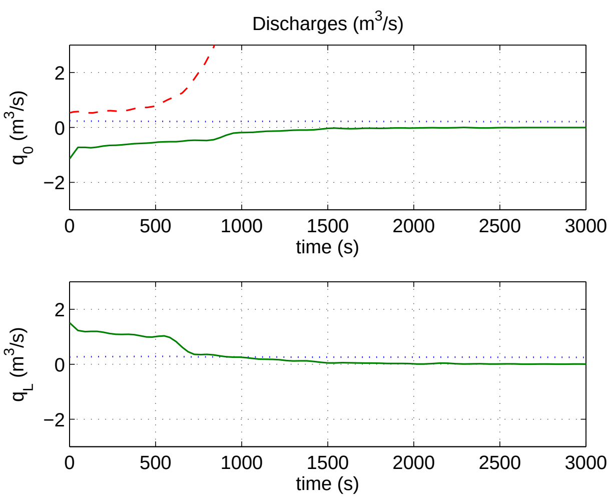

Fig. 3. Boundary discharges along time for various values of (k_0, k_L): $(-3.33, 4.63)$ (solid line), $(0.5, 0.463)$ (dotted line), $(1, -2.8)$ (dashed line)

7 CONCLUSION

The paper extends existing results on the stabilization of hyperbolic conservation laws, and proposes a frequency domain approach for the control of such systems. We have used the properties of the Callier-Desoer class of infinite dimensional transfer functions in order to derive necessary and sufficient condition for input-output stability of boundary controlled hyperbolic systems. We moreover clarified the link between input-output and exponential stability. A detailed study of proportional diagonal boundary control has provided necessary and sufficient conditions for damping in the case $\gamma = \delta = 0$, which have been extended to the general case. Simulations for boundary control of an open channel show the effectiveness of the approach. Finally, this paper demonstrates the usefulness of the classical frequency domain approach for analysis and control of distributed parameters systems represented by hyperbolic conservation laws. This preliminary work paves the way towards the study of the stability of the nonlinear Saint-Venant equations for any equilibrium regime.

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A Proof of rational decomposition

To show that the distributed matrix can be expressed as an infinite sum of simple elements, we apply the residue theorem to each element of the transfer matrix. The proof is closely related to the proof of the series decomposition of $\cot(z)$ in [15].

Let $\{\mathcal{C}_N;\, N \geq 0\}$ a series of nested contours such that there are exactly two poles p_N and p_{-N} between $\mathcal{C}_{N-1}$ and $\mathcal{C}_N$. When N is larger than k_m, the poles p_N and p_{-N} are complex conjugate.

Let us first define the function $s \mapsto f_{ij}(s) = p_{ij}(s) - \frac{a_{ij}^{(0)}}{s}$, with $a_{ij}^{(0)}$ the residue of the function $p_{ij}(s)$ in zero, where the $p_{ij}(s)$ are given by eqs. (13–16) for $i, j \in \{1, 2\}$. This function is meromorphic and can be continuously extended in $s = 0$ by $f_{ij}(0) = \frac{d}{ds}[s p_{ij}(s)]|_{s=0}$.

We apply the Cauchy residue theorem to the function $s \mapsto \frac{f_{ij}(x,s)}{z-s}$. For all $N > 1$, we have:

$$\frac{1}{2\jmath\pi} \oint_{\mathcal{C}_N} \frac{f_{ij}(s)}{z-s} ds = \sum_{k=-N, k\neq 0}^{N} \frac{a_{ij}^{(k)}}{z - p_k} - f_{ij}(z) \quad \text{(A.1)}$$

with $a_{ij}^{(k)} = \lim_{s\to p_k} (s - p_k) f_{ij}(s)$ and $\jmath^2 = -1$.

For $z = 0$, equation (A.1) leads to:

$$\frac{1}{2\jmath\pi}\oint_{\mathcal{C}_N}\frac{f_{ij}(s)}{s}ds = f_{ij}(0) + \sum_{k=-N,k\neq 0}^{N}\frac{a_{ij}^{(k)}}{p_k} \tag{A.2}$$

Subtracting (A.2) from (A.1) gives:

$$\begin{aligned} f_{ij}(z) = f_{ij}(0) &+ \sum_{k=-N,k\neq 0}^{N} a_{ij}^{(k)}\left(\frac{1}{z-p_k}+\frac{1}{p_k}\right) \\ &+ \frac{1}{2\jmath\pi}\oint_{\mathcal{C}_N} f_{ij}(s)\left(\frac{1}{z-s}+\frac{1}{s}\right)ds \end{aligned}$$

or

$$\begin{aligned} f_{ij}(z) = f_{ij}(0) &+ \sum_{k=-N,k\neq 0}^{N} a_{ij}^{(k)}\left(\frac{1}{z-p_k}+\frac{1}{p_k}\right) \\ &+ \frac{z}{2\jmath\pi}\oint_{\mathcal{C}_N}\frac{f_{ij}(s)}{s(z-s)}ds \end{aligned}$$

Now, since $|f_{ij}(s)|$ is bounded, the integral on the right hand side tends to zero as N tends to infinity.

Finally, going back to the original transfer functions, we obtain:

$$p_{ij}(s) = c_{ij} + \frac{a_{ij}^{(0)}}{s} + \sum_{k=-\infty,k\neq 0}^{\infty}\frac{s a_{ij}^{(k)}}{p_k(s-p_k)}$$

with $c_{ij} = f_{ij}(0)$, which is the result we wanted to prove.

The constants c_{ij} are given by the following expressions:

$$\begin{aligned} c_{11} &= \frac{1}{(e^{\psi}-1)^2}\left[\frac{\alpha-\beta}{\alpha\beta}\left(e^{\psi}(1-\psi)-1\right)\right. \\ &\quad \left.+\frac{\delta}{\gamma}\left(e^{2\psi}-2\psi e^{\psi}-1\right)\right] \\ c_{12} &= \frac{1}{(e^{\psi}-1)^2}\left[\frac{\alpha-\beta}{\alpha\beta}\left(1-e^{\psi}(1-\psi)\right)\right. \\ &\quad \left.+\frac{\delta}{\gamma}\left((\psi-2)e^{\psi}+2+\psi\right)\right] \\ c_{21} &= \frac{e^{\psi}}{(e^{\psi}-1)^2}\left[\frac{\alpha-\beta}{\alpha\beta}\left(e^{\psi}-1-\psi\right)\right. \\ &\quad \left.+\frac{\delta}{\gamma}e^{\psi}\left((2-\psi)e^{\psi}-(2+\psi)\right)\right] \\ c_{22} &= \frac{1}{(e^{\psi}-1)^2}\left[\frac{\alpha-\beta}{\alpha\beta}e^{\psi}\left(1+\psi-e^{\psi}\right)\right. \\ &\quad \left.+\frac{\delta}{\gamma}\left(1-e^{2\psi}+2\psi e^{\psi}\right)\right] \end{aligned}$$

with $\psi = \frac{\gamma L}{\alpha\beta}$.

B Proof of Proposition 2

The proof of the proposition is a consequence of results given in [29,30]. We explain how exponential stability can be deduced from a dissipativity argument without any assumption on the regularity of the storage function.

Let us first recall that the available storage, S_a, of a time-invariant dynamical system, Σ defined from $\mathcal{L}_2([0,t],\mathbb{R}^p)$ into $\mathcal{L}_2([0,t],\mathbb{R}^m)$, with supply rate $w(t)$, is the function from Z into $\overline{\mathbb{R}}^+$ defined by [30]:

$$S_a(z) = \sup_{\substack{z\rightarrow \\ t\geq 0}} -\int_0^t w(\tau)d\tau \tag{B.1}$$

where the supremum is taken on any interval of time $[0,t]$ with $t \in [0,\infty)$ over all motions starting in state z at time $t = 0$ under any input u belonging to $\mathcal{L}_2([0,t],\mathbb{R}^p)$.

For systems with an $\mathcal{L}_2$ gain lower than η, the supply rate is defined by

$$w(t) = \eta^2\|u(t)\|^2 - \|y(t)\|^2.$$

The main interest of the dissipativity framework is to link the behavior of the state and its input-output properties and especially characterize Lyapunov-like properties. We now state the proof of Proposition 2.

Proof Since the state-space of Σ is minimal, it is routine to deduce the following properties (see [14]): if Z is uniformly reachable from $z = 0$ then $S_a(z) \leq \eta^2\alpha_r^2\|z\|_Z^2$ for all $z \in Z$. Furthermore, if Σ is uniformly observable then $S_a(z) \geq \beta_o^2\|z\|_Z^2$ and $S_a(z(T)) - S_a(z) \leq -\beta_o^2\|z\|_Z^2$ for any $T \geq T_o$ and any $z \in Z$, where $z(T)$ is the state of the system associated to the null input and the initial condition z.

Following these preliminary results, we deduce that S_a has the following upper and lower bounds:

$$\beta_o^2\|z\|_Z^2 \leq S_a(z) \leq \eta^2\alpha_r^2\|z\|_Z^2$$

and moreover that $S_a(z(t+T)) - S_a(z(t)) \leq -\beta_o^2\|z(t)\|_Z^2$ where $T \geq T_o$. On this basis, we obtain after straightforward manipulations the following inequality:

$$S_a(z(t+T)) \leq \left(1 - \frac{\beta_o^2}{\eta^2\alpha_r^2}\right)S_a(z(t)).$$

Finally, by the dissipativity inequality, one may show that $S_a(z(t))$ is a non-increasing function of time. One therefore has $S_a(z(\tau)) \leq S_a(z(0))$ for any $\tau \in [0,T]$ and thus for any $\tau \in [0,T]$ and any $k \in \mathbb{N}$:

$$\|z(\tau+kT)\|_Z^2 \leq \left(1-\frac{\beta_o^2}{\eta^2\alpha_r^2}\right)^k \left(\frac{\eta\alpha_r}{\beta_o}\right)^2 \|z(0)\|_Z^2.$$

Let us now introduce $\rho \triangleq 1-\frac{\beta_o^2}{\eta^2\alpha_r^2}$ ($\rho < 1$ since by construction $\beta_o^2\|z\|_Z^2 \leq S_a(z) \leq \eta^2\alpha_r^2\|z\|_Z^2$) and $d \triangleq \left(\frac{\eta\alpha_r}{\beta_o}\right)^2$ (≥ 0) in order to rewrite the last inequality as

$$\|z(\tau+kT)\|_Z^2 \leq d\rho^k\|z(0)\|_Z^2$$

which implies that for any $t \geq 0$, we have

$$\|z(t)\|_Z \leq ae^{-bt}\|z(0)\|_Z$$

with $b = -\log(\rho)/(2T)$ and $a = d^{1/2}$ which corresponds to the announced exponential stability result. ■

C Proof of Corollary 2

Let us first note that:

$$\det(I-KP(s)) = \frac{f_1(s)-f_2(s)}{1-e^{(\lambda_1(s)-\lambda_2(s))L}}$$

with

$$f_1(s) = \left(1+k_0\frac{\lambda_2(s)}{s}\right)\left(1+k_L\frac{\lambda_1(s)}{s}\right)e^{(\lambda_1(s)-\lambda_2(s))L}$$
$$f_2(s) = \left(1+k_0\frac{\lambda_1(s)}{s}\right)\left(1+k_L\frac{\lambda_2(s)}{s}\right).$$

Using the asymptotic approximations (30–31), we know that for any $\varepsilon > 0$ there exists R_0 such that for any s such that $|s| > R_0$ and $\Re(s) > 0$, we have:

$$\left|\left|\frac{f_1(s)}{f_2(s)}\right| - \left|\frac{(\beta+k_0)(\alpha-k_L)}{(\alpha-k_0)(\beta+k_L)}\right| e^{-(r_1+r_2)L-\tau\Re(s)}\right| \leq \varepsilon$$

with $\tau = L\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$.

If inequality (29) is verified, there exists $\varepsilon > 0$ such that:

$$\left|\frac{(\beta+k_0)(\alpha-k_L)}{(\alpha-k_0)(\beta+k_L)}\right| e^{-(r_1+r_2)L} \leq 1-2\varepsilon$$

and then for $|s| > R_0$ we have:

$$\left|1-\frac{f_1(s)}{f_2(s)}\right| \geq \left|1-\left|\frac{f_1(s)}{f_2(s)}\right|\right| \geq \varepsilon.$$

We then conclude that there exists R_0 such that $|\det(I-KP(s))| > 0$ when condition (29) is fulfilled. ■

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